ON CHARACTERIZATION OF CONFORMAL GRADIENT VECTOR FIELD

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Abstract. In this paper we shall prove that a compact Riemannian manifold with positive constant scalar curvature is isometric to a sphere provided that it admits a nonzero conformal gradient vector field

Key words: Scalar curvature; conformal vector field; conformal gradient vector field; isometry to a sphere.

1.Introduction

Spheres have many interesting geometrical properties among the class of compact connected Riemannian manifolds. That is why, it is an important issue to classify spheres (cf. [1], [2], [4], [5], [6]). An interesting property is the existence of nonconstant functions f on $S^n(c)$ which satisfies $\nabla_X \mathrm{grad}\, f = -cfX$, where grad f is the gradient of f and ∇_X is the covarient derivative operator with respect to the smooth vector X. Obata showed that a complete connected Riemannian manifold that admits a non constant solution of this differential equation is necessarily isometric to $S^n(c)$ (cf. [5]). Deshmukh and Alsolamy [3] gave an answer for the question: " under what conditions does an n-dimensional compact and connected Riemannian manifold that admits a nonzero conformal gradient vector field has to be isometric to a sphere $S^n(c)$?", by giving certain bounds for the Ricci curvature which involves the first nonzero eigenvalue of the Laplacian operator on M. In this paper we will provide an answer to this question by restricting the scalar curvature to be positive constant as follows:

Theorem.*Let (M, g) be an n- dimensional compact connected Riemannian man- ifold of positive constant scalar curvature n(n−1)c. If M admits a non-zero conformal gradient vector field, then M is isometric to the n-sphere $S^n(c)$.*

2.Preliminaries

Let (M,g) be a Riemannian manifold with Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. A vector field $X \in \mathfrak{X}(M)$ is said to be conformal if it satisfies

(2.1) $\mathfrak{L}_X g = 2\boldsymbol{\varphi} g$

For a smooth function $\boldsymbol{\varphi}: M \to R$, where $\mathfrak{L}_X$ is the Lie derivative with respect to X. If u = grad f is the gradient of a smooth function f on M and u is a conformal vector field, then it follows from

(2.1) that a conformal vector field u satisfies

(2.2) $\nabla_X u = \varphi X$, $X \in \mathfrak{X}(M)$

For a sphere $S^n(c)$, there exists a non constant function $\varphi \in C^\infty(S^n(c))$ which satisfies

(2.3) $\nabla_X \nabla\varphi = -c\varphi X$

where $\nabla\varphi$ is the gradient of φ and ∇_X is the covariant derivative operator with respect to the smooth vector X.

The following result is an immediate consequence of the equation (2.2):

Lemma 2.1 *Let u be a conformal gradient vector field on a compact Riemannian manifold (M, g). Then, for* $\varphi = n^{-1} \operatorname{div} u$,

$$\int_M \varphi\, dv = 0$$

For a smooth function f on M, define an operator A $:\mathfrak{X}(M) \to \mathfrak{X}(M)$ by $AX = \nabla_X \nabla f$, where ∇f is the gradient of f . The Ricci operator Q is a symmetric operator defined by

$Ric(X,Y) = g(Q(X),Y)$, $X,Y \in \mathfrak{X}(M)$

where *Ric* is the Ricci tensor of the Riemannian manifold, and hence from the definition of the operator A we have the following relation

(2.4) $R(X, Y)\nabla f = (\nabla A)(X, Y) - (\nabla A)(Y, X)$

where $(\nabla A)(X,Y) = \nabla_X AY - A(\nabla_X Y)$, $X,Y \in \mathfrak{X}(M)$ Note that using (2.4) we have the following lemma which gives an important property of the operator A.

Lemma 2.2 *Let (M,g) be Riemannian manifold and f be a smooth function on M. Then the operator A corresponding to the function f satisfies*

$$\sum_{i=1}^{n} (\nabla A)(e_i, e_i) = \nabla(\Delta f) + Q(\nabla f)$$

Where $\{e_1, e_2, \ldots, e_n\}$ *be a local orthonormal frame on M.*

Lemma 2.3 *Let u be a conformal gradient vector field on an n- dimensional Riemannian manifold (M,g). Then the operator Q satisfies*

$$Q(u) = -(n-1)\nabla\varphi,$$

Where $\nabla\varphi$ *is the gradient of the smooth function* $\varphi = = n^{-1} div\, u$.

3.Proof of the Theorem

For an n- dimensional compact connected Riemannian manifold (M,g) of positive constant scalar curvature $S = n(n-1)c$, we have that $\Delta\varphi = -nc\varphi$ and from Lemma 2.3 that $\Delta f = n\varphi$. These two relations imply $\Delta\varphi = = -c\Delta f$ that is $\Delta(\varphi + cf) = 0$. Thus $\varphi = -cf + \alpha$, where α is a constant. Consequently $\nabla\varphi = -c\nabla f$, which gives $\nabla_X \nabla\varphi = -c\nabla_X \nabla f = -c\varphi X$ that is φ satisfies the Obata's differential equation. We claim that $\varphi$is not constnat. If $\varphi$is a constant, it will imply that f is a constant which in turn will imply that u = 0 and that leads to a contradiction as the statement of the Theorem requires that u is nonzero vector field. Hence by Obata's theorem we get that M is isometric to the n-sphere $S^n(c)$.

Remarks: The compactness condition in the theorem essential, as for the Riemannian manifold (R^n, g), where g is the Riemannian metric defined by

$g = \frac{1}{1+\|x\|^2}\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the Euclidean metric on R^n, choosing u to be the position vector field, the scalar curvature S of (R^n,g) is a positive constant but the manifold is not isometric to a sphere $S^n(c)$.

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